THE GEOMETRIES ASSOCIATED WITH A CERTAIN SYSTEM OF CREMONA GROUPS*

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Introduction. A geometry is defined by a class of elements S and a group of transformations G operating in a one-to-one reciprocal way on the elements of S and transforming S into itself. The geometry (S, G) then consists of such properties of figures (i. e., combinations of elements of S) as remain invariant under the transformations of G. This is in brief Klein's classical principle.†

The most familiar geometries having the point as the primary element of space which have received systematic study are:

(1) The geometries defined by the general *projective groups* in one, two, . . . , n dimensions, and their subgroups.

(2) The geometries defined by the *inversion groups* in two, three, and to some extent n dimensions.

The object of the present paper is to call attention to a system of groups $G_2, G_3, \cdots, G_n$ of Cremona transformations in point spaces $S_2, S_3, \cdots, S_n$ of two, three, . . . , n dimensions and to develop elementary properties of the corresponding geometries. This system of groups includes for the case $n = 2$ the group G_2 of direct circular transformations in the plane; the corresponding plane geometry is then closely related to the ordinary inversion geometry in the plane. The associated geometries in three or more dimensions form, therefore, *a direct extension of the inversion geometry in a plane*, different from the inversion geometries in the spaces of three or more dimensions. The elementary curves in the geometry (S_2, G_2) are the conics through two fixed points in a projective plane (in the ordinary orientation of the inversion geometry these points are the circular points at infinity so that the conics referred to are circles). In the geometries (S_3, G_3) the elementary

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curves are twisted cubics; in the geometries (S_n, G_n) they are normal curves of order n in S_n.

Definition of groups $G_2, G_3, \cdots, G_n$. It will be sufficient to define in detail the groups G_2 and G_3. The extension to G_n will then be sufficiently obvious.

The Group G_2. Let U_1, U_2 be the distinct centers of two pencils of lines in the (real or complex) projective plane Σ_2. Every point P in the plane not on the line $U_1 U_2$ determines and is determined by a unique line in each of the pencils U_1, U_2. The point P is then determined by the two lines l_1 (through U_1) and l_2 (through U_2) and we may write $P = (l_1, l_2)$. Let π_1 be any projectivity in the pencil U_1 and π_2 any projectivity in the pencil U_2 and let $\pi_1(l_1) = l_1'$ and $\pi_2(l_2) = l_2'$. The transformation $\pi = (\pi_1, \pi_2)$ is then defined as the transformation which transforms $P = (l_1, l_2)$ into $P' = (l_1', l_2')$. π is a transformation of G_2, and G_2 consists of all such transformations π which can be obtained by varying the *component projectivities* π_1, π_2. G_2 is a group of (in general) quadratic transformations which transforms into itself transitively the system of all conics through U_1 and U_2.

The Group G_3. Let u_1, u_2, u_3 be the nonconcurrent axes of three pencils of planes in projective space Σ_3. Every point P in Σ_3 (not on the quadric surface determined by u_1, u_2, u_3) determines a unique plane in each of the pencils u_1, u_2, u_3. The point P is then determined by the three planes α_1 (through u_1), α_2 (through u_2), α_3 (through u_3) and we may write $P = (\alpha_1, \alpha_2, \alpha_3)$. Let π_1 be a projectivity in the pencil u_1, π_2 a projectivity in the pencil u_2, and π_3 a projectivity in the pencil u_3; and let $\pi_i(\alpha_i) = \alpha_i'$ $(i = 1, 2, 3)$. The transformation $\pi = (\pi_1, \pi_2, \pi_3)$ is then defined as the transformation which transforms $P = (\alpha_1, \alpha_2, \alpha_3)$ into $P' = (\alpha_1', \alpha_2', \alpha_3')$. π is in general a cubic transformation and is a transformation of G_3. G_3 consists of all such transformations π which can be obtained by varying the *component projectivities* π_1, π_2, π_3. G_3 transforms into itself transitively the system of all twisted cubics having u_1, u_2, u_3 as bisecants.

The group $G_n (n > 3)$ is defined similarly in Σ_n by means of projectivities in n pencils of Σ_{n-1}'s. They are groups of Cremona transformations of order n (in general), which transform transitively into themselves certain systems of normal curves of order n in Σ_n.

It should be noted that for $n > 2$, G_n has several distinct forms. Thus G_3 has four distinct forms according to the figure formed by the lines u_1, u_2, u_3. These lines may be skew to each other (G_3^1), or two only may intersect (G_3^2), or one of the lines may intersect each of the other two without their being coplanar (G_3^3), or finally u_1, u_2, u_3 may form a triangle (G_3^4). The cubic transformations thus defined have been extensively studied by Noether and

others.* The groups of such transformations have, however, received but little attention hitherto.†

In what follows explicit reference is generally made only to the case $n = 3$. Most of our theorems, however, apply either without any change or with very obvious changes to all cases, $n \geqq 2$. The following notational scheme enables us readily to assign the proper generality to the later results. For the case $n = 2$ there is only one group G_2 while for the case $n = 3$ there are four groups $G_3^1, G_3^2, G_3^3, G_3^4$: the symbol G_3 is used to represent any one of these last four groups. Similarly G_n represents any one of the groups possible for a general n; while, when the number of dimensions is not significant, we shall simply use G to represent any one of the whole system of groups.

The space S. The group G_3 has been defined in terms of, and as acting on, the (real or complex) projective point space Σ_3 of three dimensions. It does not, however, operate on the points of Σ_3 in a one-to-one reciprocal way. Our first problem is, then, to define a space S_3, on the primary elements of which G_3 operates in a one-to-one reciprocal way. The definition of such a space is very simple, for we need only keep in mind the fact that G_3 does operate in a reciprocally one-to-one way on the triples of planes $\alpha_1, \alpha_2, \alpha_3$. We may therefore simply define these *plane-triples* as the "points" of our space S_3.‡ However for the sake of greater intuitional satisfaction, we prefer to replace these plane triples by "points," proper and improper, determined by them in a one-to-one reciprocal way.

We have first as *proper points* of S_3, all the points of Σ_3 which are not on the quadric surface $(u_1 u_2 u_3)$.§

We have secondly, *improper points*, in general as follows:

(1) *Straight lines* when the planes of the plane-triple $(\alpha_1, \alpha_2, \alpha_3)$ intersect in a line. These lines are the lines meeting u_1, u_2, and u_3.

(2) *Point-planes*, consisting of a point on u_i $(i = 1, 2, 3)$ with a plane through u_i. These occur when two of the planes $(\alpha_1, \alpha_2, \alpha_3)$ meet on u_i.

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* Noether, *Über die eindeutigen Raumtransformationen, insbesondere in ihrer Anwendung auf die Abbildung algebraischer Flächen*, M a t h e m a t i s c h e   A n n a l e n, Bd. 3 (1871), S. 547–580.

Doehlemann, *Über eine einfache eindeutige Raumtransformation dritter Ordnung*, M ü n - c h e n e r   B e r i c h t e, Bd. 24 (1894), S. 41–50.

Doehlemann, *Geometrische Transformationen*, Teil (2), S. 308.

Sturm, *Die Lehre von den Geometrischen Verwandtschaften*, IV, § 127.

† Fano, *I gruppi di Jonquières generalizzati*, T o r i n o   M e m o r i e (2), vol. 48 (1898), p. 221.

Hudson, *The (3, 3) Transformations in Space*, P r o c e e d i n g s   o f   t h e   L o n d o n   M a t h e m a t i c a l   S o c i e t y, 1913.

‡ For G_n we should define the " points " of S_n as the corresponding n-ples of Σ_{n-1}'s.

§ This quadric degenerates in the cases G_3^2, G_3^3 into two planes; in case of G_3^4 into a single plane.

When two of the lines u_1, u_2, u_3 intersect, other forms of improper points arise as will be seen by a direct consideration of the cases in question.

To define completely the spaces S_3^1, S_3^2, S_3^3, S_3^4 associated respectively with G_3^1, G_3^2, G_3^3, G_3^4, it is only necessary to list the improper points associated in each case with the proper points to form the space S_3 in question. We give below such a definition for S_3^1 because of its simplicity, and for S_3^4 as it is in some respects the most interesting.

The space S_3^1 of G_3^1. (The lines u_1, u_2, u_3 are skew to each other.)

(1) *Proper points:* All the points of the projective Σ_3 not on the quadric $(u_1 u_2 u_3)$.

(2) *Improper points:*

(i) All the *lines* meeting u_1, u_2, u_3.

(ii) All the *point-planes* consisting of a point P on u_i ($i = 1, 2, 3$) together with any plane through u_i distinct from the tangent plane to the quadric $(u_1 u_2 u_3)$ at P.

If the projective Σ_3 is real, there are ∞^1 improper points of the *first kind*, and three systems of ∞^2 improper points of the *second kind*.

The space S_3^4 of G_3^4. (The lines u_1, u_2, u_3 form a triangle.)

(1) *Proper points:* All points of the projective Σ_3 not on the plane of the triangle.

(2) *Improper points:*

(i) All *linear elements* consisting of a vertex U_i ($i = 1, 2, 3$) of the triangle $u_1 u_2 u_3$ and a line through U_i, not in the plane of u_1, u_2, u_3.

(ii) All *line-planes* consisting of a line u_i ($i = 1, 2, 3$) and any plane through u_i distinct from the plane of $u_1 u_2 u_3$.

(iii) The *plane* of the triangle $u_1 u_2 u_3$.

If S_3 is real, there are three systems of ∞^2 improper points of the *first kind*, three systems of ∞^1 improper points of the *second kind*, and one improper point of the *third kind*.

Throughout the remainder of the paper unless otherwise specified the word "point" will refer to an element, proper or improper, of the space S associated with the group G.

Analytic representation of G. The associated hyper-complex number system. If we establish a projective *scale* in each of the pencils u_i ($i = 1, 2, 3$) by assigning the numbers $0, 1, \infty$ to any three distinct planes through u_i,[*] then every point P in S_3 may be denoted by three coördinates (x_1, x_2, x_3), x_i being the coördinate of the plane through P in u_i in the scale just referred to. The transformations of the group G_3 are then given by the relations:

$$(1) \qquad x_i' = \frac{a_i x_i + b_i}{c_i x_i + d_i}, \qquad a_i d_i - b_i c_i \neq 0 \qquad (i = 1, 2, 3).$$

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* Veblen and Young, *Projective Geometry*, Vol. 1, Boston, 1910, p. 141.

If $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ are any two points of S_3 (neither of which is on a plane ∞) a hypercomplex algebra N_3* is defined by the relations,

(2)
$$A + B = (a_1 + b_1, \ a_2 + b_2, \ a_3 + b_3),$$
$$A \cdot B = (a_1 b_1, \ a_2 b_2, \ a_3 b_3).$$

It is at once evident from this definition that N_3 satisfies the ordinary associative, commutative, and distributive laws as to addition and multiplication.†
The nil-factors (divisors of zero) are the elements (x_1, x_2, x_3) in which one or more of the x_i $(i = 1, 2, 3)$ are zero. Moreover it should be noted that the elements of the form (x, x, x) form a sub-algebra R_3 of N_3 isomorphic with the algebra of ordinary real or complex numbers according as x ranges over the system of real or complex numbers. The numbers of R_3 may be conveniently designated as the *scalars* of N_3.

In the terms of this algebra, the transformations of G_3 are given by the relations:

(3) $$X' = \frac{AX + B}{CX + D} \qquad (AD - BC \neq \text{a nil factor}),$$

where A, B, C, D represent any elements of N_3.

The extension of these results to any group G_n is obvious.‡

Invariant of G. Cross-ratio. Associated with every ordered set of four points

$$A = (\alpha_1, \alpha_2, \alpha_3), \qquad B = (\beta_1, \beta_2, \beta_3),$$
$$C = (\gamma_1, \gamma_2, \gamma_3), \qquad D = (\delta_1, \delta_2, \delta_3)$$

are three cross-ratios $\rho_i = R_f(\alpha_i \beta_i, \gamma_i \delta_i)$ $(i = 1, 2, 3)$. The *cross ratio symbol* $\rho = (\rho_1, \rho_2, \rho_3) = R_f(AB, CD)$ is then invariant under G_3, and is hence fundamental in the geometry (S_3, G_3). It will be noted that the cross-ratio symbol ρ is an element of the hypercomplex algebra N_3. This close connection of (S, G)§ with the projective geometry in a one-dimensional

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* In papers by J. W. Young, *A New Formulation for General Algebra*, read before the American Mathematical Society, Dec. 28, 1911, and Apr. 26, 1913 (abstracts in B u l l e t i n, vol. 18, Feb., 1912; vol. 19, July, 1913), the groups here discussed were discovered as a natural result of the investigations of these papers.

† The algebra N_3 is not invariant under G, that is to say if $A + B = C$, and if an operation of the group sends A, B, C into A', B', C' then, in general, $A' + B' \neq C'$.

‡ The representation of G in the form (3) exhibits strikingly the analogy of G with the group G_2 of direct circular transformations in a plane. The corresponding algebra N_2 is equivalent to the algebra of ordinary complex numbers if properly orientated in the *real S_2*. The representation of the real circular transformations in a plane by linear fractional transformations on a complex variable is well known.

§ Reference may be made to a paper by J. W. Young, *The Geometry of Chains on a Complex Line*, A n n a l s   o f   M a t h e m a t i c s, series 2, vol. 11, pp. 33–48, 1909–10. The methods and the results of this paper and the present one bear a close analogy. Indeed, the geometries (S, G) are in a sense geometries on a hypercomplex line.

form at once suggests a method of approach to the geometry (S, G). Certain conceptions and the associated terminology of one-dimensional projective geometry may at once be carried over. For example, two pairs of points AB, CD of S_3 form a harmonic set $H(AB, CD)$ if the sets of planes $(\alpha_i \beta_i, \gamma_i \delta_i)$ $(i = 1, 2, 3)$ are harmonic: i. e., if $R(AB, CD) = (-1, -1, -1)$. It is natural to designate a scalar (x, x, x) of N_3 by the number x. With this notation we may say that AB, CD form a harmonic set if

$$R(AB, CD) = -1.$$

The fundamental theorem of G. Let us call two points $A = (\alpha_1, \alpha_2, \alpha_3)$ and $A' = (\alpha_1', \alpha_2', \alpha_3')$ of S_3 *co-axial* if they determine the same plane in any one of the pencils u_i, i. e., if either $\alpha_1 = \alpha_1'$, or $\alpha_2 = \alpha_2'$, or $\alpha_3 = \alpha_3'$.*

THEOREM 1. *If A, B, C and A', B', C' are any two sets of three points of S, such that no two points of either set are co-axial, then there exists one and only one transformation π of G, such that $\pi(ABC) = A'B'C'$.*

This theorem is a direct consequence of the definition of G and the fundamental theorem of projective geometry.

The simple curves of (S, G). Let A, B, C be any three points of S_3 no two of which are co-axial, and consider the locus of a point P such that $R(AB, CP)$ is a scalar $\lambda = (\lambda, \lambda, \lambda)$ of N_3, where λ is any real or complex variable. This locus may also be defined as the locus of points determined by the triples of homologous planes in the projectivities

$$\alpha_1 \beta_1 \gamma_1 \barwedge \alpha_2 \beta_2 \gamma_2 \barwedge \alpha_3 \beta_3 \gamma_3.$$

A curve thus defined we will call a *C-curve*. In view of the following theorems, which are immediate consequences of the definition and Theorem 1, the C-curves are *simple one dimensional elements* of the geometry (S, G).

THEOREM 2. *If A, B, C are any three points of S, no two of which are co-axial, there exists one and only one C-curve containing A, B, and C.*

The C-curve through A, B, C will be denoted by $|ABC|$.

THEOREM 3. *Any two C-curves are equivalent under the group G.*

The C-curves in S_3 are, in general, twisted cubics (in Σ_3) having the axes u_1, u_2, u_3 as *bisecants* and any such twisted cubic is a C-curve.† However

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* In S_n, $A = (\alpha_1, \alpha_2, \alpha_3, \cdots \alpha_n)$ and $A' = (\alpha_1', \alpha_2', \alpha_3', \cdots \alpha_n')$ are co-axial if one of the relations $\alpha_i = \alpha_i'$ holds.

† That the locus of the point of intersection of the homologous planes of three projective axial pencils is in general a twisted cubic having the axes of the pencils as bisecants, seems to have been first proved by Chasles, C o m p t e s R e n d u s, vol. 45 (1857), p. 189 and J o u r n a l d e M a t h é m a t i q u e s, 2d ser., vol. 2 (1857), p. 397. See also Reye, *Geometrie der Lage*, vol. 2, pp. 91–93; Sturm, *Geometrische Verwandtschaften*, vol. I, p. 306. In S_n the C-curves are normal curves of order n in Σ_n, having as $(n-1)$-secants the " axes " of the generating pencils of Σ_{n-1}'s, and every such normal curve is a C-curve: see Bertini, *Introd. alla geometria proiettiva degli iperspazi*, p. 275.

these cubics degenerate under certain conditions into conics and straight lines. It is quite unnecessary for the further development of this paper, to describe these curves in detail; for, all later theorems are consequences merely of the definitions and Theorems 1, 2, 3. It seems desirable nevertheless to list the C-curves completely, at least for S_3^1 and S_3^4.

The class of C-curves in S_3^1 consist of: (1) All the *twisted cubics* of Σ_3 having u_1, u_2, u_3 as bisecants. These are characterized in S_3^1 by the fact that none of them contains an improper point of the first kind. (2) All the *conics* of Σ_3 not on the quadric $u_1 u_2 u_3$ but meeting u_1, u_2, and u_3 each once. These are characterized in S_3^1 by the fact that each such C-curve contains one and only one improper point of the first kind. (3) All the *straight lines* of Σ_3 not on the quadric $u_1 u_2 u_3$ nor meeting u_1, u_2, u_3. These are characterized by the fact that each such C-curve contains two and only two distinct improper points of the first kind. (4) The *locus of all improper points of the first kind.*

The class of C-curves for S_3^4 consists of: (1) All *twisted cubics* of Σ_3 containing the vertices U_1, U_2, U_3 of the triangle u_1, u_2, u_3. These are characterized in S_3^4 by the fact that every such C-curve contains three improper points of the first kind and no other improper points. (2) All *conics* of Σ_3 not on the plane u_1, u_2, u_3 but passing through U_i ($i = 1, 2, 3$) and meeting the opposite side of the triangle. These are characterized in S_3^4 by the fact that each such conic contains an improper point of the first kind, one of the second kind and no other improper points. (3) All *straight lines* of S_3 not through a side of the triangle $u_1 u_2 u_3$. These are characterized in S_3 by the fact that each such C-curve contains the improper point of the third kind and no other improper point.*

A fundamental property of a C-curve, which follows at once from the definition, is given by

THEOREM 4.   *If A, B, C are three distinct points of a C-curve, the harmonic conjugate of A with respect to B and C is on the curve.*

Properties of G.   The projective structure of the group G makes it possible to infer numerous properties of the group almost immediately. The enumeration of subgroups and their properties, questions of the conjugacy of trans-

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* *Metric orientation.* In the usual orientation of the real geometry (S_2, G_2) the two points $U_1 U_2$ coincide with the circular points at infinity in the plane Σ_2. The system of C-curves (the conics through U_1, U_2) then appear as the system of circles (and straight lines) in S_2. An analogous orientation of the geometry (S_3^4, G_3^4) in real Σ_3 may be obtained by letting the triangle u_1, u_2, u_3 consist of two conjugate tangents to the absolute conic at infinity and their real chord of contact. If U_i be the real intersection of these tangents, and if we choose the direction determined by U_i to be vertical, the system of real C-curves consists of (1) The twisted cubics on all right circular cylinders whose axes are vertical. (2) The rectangular hyperbolas one of whose asymptotes is vertical. (3) The straight lines which are neither vertical nor horizontal.

formations, etc., while simple, we pass over as not of immediate concern in the present paper. The following results are needed, however, in what follows.

THEOREM 5. *Every transformation of G transforms a harmonic set of points into a harmonic set of points.*

THEOREM 6. *Two transformations of G with the same double points are commutative.*

The involutions, I, of G (i. e., transformations I such that $I^2 = 1$, $I \neq 1$) are of special interest. Since every component involution in the generating pencils has two distinct double planes in (complex) Σ_3, it is clear that every involution I of G_3 has in general $2^3 = 8$ double points in complex S_3. These eight double points consist of four pairs, the two points of each pair not being co-axial. Such a pair of non-co-axial double points will be called a pair of *conjugate double points** of I and will be denoted by $M_i\,N_i\,(i = 1, 2, 3, 4)$.†
We have then,

THEOREM 7. *In an involution I of G any point A and its image A' are harmonic with respect to any pair of conjugate double points $M_i\,N_i$ of I. Symbolically $I(A) = A' = H(A, M_i\,N_i)$.*

THEOREM 8. *An involution I of G is completely determined by two pairs of homologous points; and, in particular, by a pair of conjugate double points.*

THEOREM 9. *Any transformation π of G is the product of two involutions of G.*

THEOREM 10. *If A, A' is a pair of homologous points of an involution I, the involution having A, A' for double points is commutative with I.*

COROLLARY. *The product of two involutions with conjugate double points A, A' and B, B', respectively, transforms into itself the involution in which A, A' and B, B' are homologous pairs.*

Invariant Curves. It will be supposed in what follows that S is complex.

THEOREM 11. *If a C-curve through $M_i\,N_i$ and one through $M_j\,N_j\,(i \neq j)$ [$M_i\,N_i$ and $M_j\,N_j$ being conjugate double points of an involution I] have one point in common they have a second point in common.*

Let C and C' be the C-curves and let P be their common point.

Since the harmonic conjugate of P is on both C and C' the curves have a second point in common.

THEOREM 12. *Every C-curve containing a pair of conjugate double points $M_i\,N_i$ of an involution I is invariant under I.*

Let C be any C-curve containing the double points $M_i\,N_i$ and let A be any

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* An involution I of G_n has in general 2^n double points in (complex) S_n consisting of 2^{n-1} pairs of non-co-axial points, i. e., *conjugate double points.*

† Intuitively, the situation is clarified by thinking of the eight double points as the eight corners of a box. Conjugate double points are diagonally opposite corners.

point of $C = |M_i N_i A|$. Then $I(A) = A' = H(A, M_i N_i)$. Hence A' is a point of C.

Theorem 13. *Every C-curve invariant under an involution I passes through a pair of conjugate double points of I.*

Corollary. *Through every point in S_3 distinct from $M_i N_i (i = 1, 2, 3, 4)$ pass four and only four C-curves invariant under I.*

Theorem 14. *If a C-curve C not passing through a given pair of conjugate points $M_i N_i$ of an involution I meets two C-curves through $M_i N_i$ in two pairs of points harmonic with regard to $M_i N_i$ then C passes through a pair of double points $M_j N_j (i \neq j)$.*

Let A, A' and B, B' be the two pairs of points. Then $H(M_i N_i, AA')$ and $H(M_i N_i, BB')$ gives $M_i N_i$ as the double points of the involution $I(AA', BB')$. Therefore $|AA'B| = |A'AB'|$ which is an invariant C-curve and thus by Theorem 12 must pass through the points $M_j N_j$.

Theorem 15. *If some C-curve is left invariant by a non-involutoric transformation π of G, then there is one and only one C-curve invariant under π through every point P of S not co-axial with a double point of π.*

Let C be the C-curve invariant under π, and let A be any point of C not a double point of π. Let π_1 be the transformation of G having the same double points as π and such that $\pi(A) = P$. Let $\pi_1(C) = C'$. Then

$$\pi_1 \pi \pi_1^{-1}(C') = C'$$

but $\pi_1 \pi \pi_1^{-1} = \pi$, therefore $\pi(C') = C'$, which proves that there is at least one C-curve through P invariant under π. Now suppose that there are two distinct C-curves through P invariant under π and let

$$\pi(P) = P', \qquad \pi(P') = P''.$$

The curve C' being invariant, gives P, P', P'' as points of C'. If $P'' \neq P$ these three points determine C' uniquely: if $P'' = P$, π is an involution and this contradicts the hypothesis.

Theorem 16. *If the transformation π of G is parabolic (i. e., has only one double point) then through every point in S non-co-axial with the double point, passes one and only one C-curve invariant under π and every such curve passes through the double point.*

Let A be any point in S and $\pi(A) = A'$, $\pi(A') = A''$. Then, if M is the double point, (MA', AA'') is a harmonic set. But $\pi(AA'M) = A'A''M$. Therefore, there is a C-curve through A invariant under π. That there is only one follows from Theorem 15.

Tangent C-Curves. Two C-curves are said to be tangent to each other provided there exists a parabolic transformation π of G which leaves them

both invariant.* Two such C-curves will have only one point M in common (Theorem 16). The point M is called the point of contact.

The relation of tangency is clearly an invariant of the geometry $[S, G]$.

THEOREM 17. *If M is a point on a C-curve C and P any point non-co-axial with M and not on C, then there exists one and only one C-curve through P and tangent to C at M.*

For let π be any parabolic transformation of G with double point M, and let $\pi(P) = P'$. $|PP'M|$ is then the C-curve sought.

Orthogonal C-Curves. A pair of non-co-axial points M, N of S_3 determines uniquely three other pairs of non-co-axial points, namely the other three pairs of conjugate double points of the involution of which M, N are one pair of conjugate double points. In real S_3 if M, N are real, the other three pairs may all be imaginary. It seems convenient, therefore, to introduce the following terminology derived by obvious analogy from the geometry (S_2, G_2).

A C-curve C is said to be *about* two non-co-axial points M, N, if it passes through the points of one of the three pairs of non-co-axial points determined by M, N. The points M, N are then said to be *conjugate* with respect to C.

The relation here defined between a C-curve and certain pairs of points is clearly an invariant under G_3, as expressed by the following theorem:

THEOREM 18. *If a C-curve C is transformed by a transformation π of G into C', any pair of points conjugate with respect to C is transformed by π into a pair of points conjugate with respect to C'.*

A C-curve C_2 is said to be *orthogonal* to a C-curve C_1, if C_2 contains two points conjugate to C_1.†

THEOREM 19. *If C_2 is orthogonal to C_1, C_1 is orthogonal to C_2.*

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* Two C-curves tangent to each other as here defined are tangent in the ordinary sense, and conversely. For the coördinate system described on p. 236 may be so chosen that any point M is $(0, 0, 0)$ and that any C-curve through M is given by $x = y = z$. Any other C-curve through M is then given by the equations

(1) $$\frac{ax}{cx + d} = \frac{a'y}{c'y + d'} = \frac{a''z}{c''z + d''}, \qquad aa'\,a''\,dd'\,d'' \neq 0.$$

The conditions that these intersect in a second point $T = (t, t, t)$ $(t \neq 0)$ are

$$\frac{a}{ct + d} = \frac{a'}{c't + d'} = \frac{d''}{c''t + d''},$$

and that they be tangent in the ordinary sense at M $(t = 0)$ are

$$\frac{a}{d} = \frac{a'}{d'} = \frac{a''}{d''}.$$

But these are precisely the conditions that (1) can be obtained from $x = y = z$ by a parabolic transformation with double point M.

† It should be noted that orthogonality, as here defined, is not in general equivalent to orthogonality in the ordinary sense.

THEOREM 20. *Any pair of orthogonal C-curves is transformed by a transformation of G into a pair of orthogonal C-curves.*

This theorem establishes orthogonality as an invariant of G.

THEOREM 21. *If a transformation π of G leaves each of two orthogonal C-curves invariant, π is an involution.*

This follows from the proof of Theorem 15.

THEOREM 22. *Through two distinct points A, B on a C-curve C in S_3, pass three and only three C-curves orthogonal to C.*

For, let M, N be any pair of points on C harmonic with A, B: and let $M_i N_i$ $(i = 1, 2, 3)$ be the three other pairs of double points of the involution I having M, N for double points. Then we have $H(M_i N_i, AB)$ and the three curves $|ABM_i|$ $(i = 1, 2, 3)$ are orthogonal to C. That no other C-curves orthogonal to C pass through A, B will follow, if we show that the three C-curves orthogonal to C obtained as above from any other two points M', N' of C, harmonic with AB, coincide with those obtained from M, N. To this end we will show that if $ABMM'$ are on C, then $ABM_1 M_1'$ will be on the same C-curve. The following notational scheme is used:

$$M = (u_1, u_2, u_3), \qquad N = (v_1, v_2, v_3), \qquad M' = (u_1', u_2', u_3'),$$

$$N' = (v_1', v_2', v_3'), \qquad A = (\alpha_1, \alpha_2, \alpha_3), \qquad B = (\beta_1, \beta_2, \beta_3):$$

finally $M_1 = (v_1, u_2, u_3)$ and $M_1' = (v_1', u_2, u_3)$. The condition that A, B, M, M' are on C is

$$R(\alpha_1 \beta_1, u_1 u_1') = R(\alpha_2 \beta_2, u_2 u_2') = R(\alpha_3 \beta_3, u_3 u_3').$$

We must show that

$$R(\alpha_1 \beta_1, v_1 v_1') = R(\alpha_2 \beta_2, u_2 u_2') = R(\alpha_3 \beta_3, u_3 u_3')$$

or simply that

$$R(\alpha_1 \beta_1, u_1 u_1') = R(\alpha_1 \beta_1, v_1 v_1').$$

But this equality follows directly from the relations $H(\alpha_1 \beta_1, u_1 v_1)$ and $H(\alpha_1 \beta_1, u_1' v_1')$. The theorem is clearly valid in any geometry (S, G), if the number of C-curves through A, B orthogonal to C is replaced by $2^{n-1} - 1$.

Orthogonal Systems of C-Curves. It is now natural to generalize the well-known orthogonal systems of circles in (S_2, G_2). These are of two kinds. (1) The system of all circles through two distinct points A, B and the system of all circles about the two points A, B. (2) The system of all circles tangent at a point M to a given circle through M and the system of all circles tangent at M to a circle through M orthogonal to a circle of the first system.

In (S_3, G_3) we have analogously:

THEOREM 23.   *Two non-co-axial points M, N of S_3 determine four systems of C-curves, viz., the system of all C-curves through M, N and the three systems of C-curves about M, N.   Any C-curve of one of these four systems is orthogonal to every C-curve of the other three systems.*

THEOREM 24.   *Given four mutually orthogonal C-curves C_1, C_2, C_3, C_4 through M, P.   The four systems of C-curves consisting of all the C-curves tangent to C_i ($i = 1, 2, 3, 4$) at M are such that any C-curve of one system is orthogonal to every C-curve of the other three systems.*

In (S_n, G_n), analogous theorems hold, the number of mutually orthogonal systems of C-curves being 2^{n-1}.

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